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On the existence of solutions for a system of difference equations with non-monotone nonlinearity

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ABSTRACT

We derive a dual variational method in order to obtain the existence of a bounded solution to a certain nonlinear system with non-monotone nonlinearities. We consider such a system representing a difference equation arising from evaluating some Dirichlet boundary value problems.

1. Introduction

Variational methods (critical point theory) have lately been extensively applied in showing the existence of solutions for difference equations of certain types. In the main, the mountain pass methodology is applied, the linking theorem is involved and a Palais–Smale type condition is assumed; see [1–6] among other papers. For example in [4], difference equations arising from evaluating Dirichlet boundary value problems are considered. Such problems can often be put into the following form:

$$Au = \lambda f(u), \quad u \in R^N \tag{1.1}$$

for a real symmetric $N \times N$ matrix A and a continuous function $f : R^N \to R^N, f(x) = [f_1(x_1), \dots, f_N(x_N)]^T$, where $x_i \in R$. For example, let us consider the difference equation

$$\Delta^2 u_{k-1} = \lambda g(k, u_k), \quad k \in \{1, 2, \dots, n\} \tag{1.2}$$

with a positive parameter λ and a continuous nonlinearity $g : [0, 1] \times R \to R$ and which is subject to the conditions

$$u_0 = u_{n+1} = 0.$$

Such a difference equation may arise from evaluating the Dirichlet boundary value problem

$$\frac{d^2}{dt^2} u = \lambda g(t, u), \quad 0 < t < 1, \qquad u(0) = u(1) = 0.$$

With a positive definite $N \times N$ real symmetric matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}_{N \times N}$$

problem (1.2) can be written in the form (1.1).

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Our aim here is to relax the assumptions that f_i is nondecreasing for each $i = 1, 2, \dots, N$ which we have made in [7] and to derive a suitable version of the dual variational method in this new setting. Such an assumption yields for example that the primitive of f, i.e. such a function $F : R^N \to R$ that $\frac{\partial F(x)}{\partial x_i} = f_i(x_i)$, is a convex continuous function. In this work we try to relax the convexity of F. Instead we assume:

A1 $f_i : R \to R$ for $i = 1, 2, \dots, N$ is a C^1 function.

In that case a second-order convexity test reveals that on any interval $[-d, d]$ with $d > 0$ a function $x \to \sum_{i=1}^N F_i(x_i) + \frac{1}{2}\sum_{i=1}^N \max_{y\in[-d,d]} |\frac{df_i(y)}{dx_i}| x_i^2$ is convex and of class C^2. We denote, for a fixed $d > 0$, $g_i(x_i) = \max_{y\in[-d,d]} |\frac{df_i(y)}{dx_i}| x_i$ and so $g(x) = [g_1(x_1), \dots, g_N(x_N)]^T$. Thus for a fixed $d > 0$ we can rewrite problem (1.1) as follows:

$$Au + \lambda g(u) = \lambda h(u), \quad u \in R^N \tag{1.3}$$

where we write $h(u) = f(u) + g(u)$ for convenience and H stands for the primitive of h; $H : R^N \to R^N$ is a C^2 convex function.

In our research we use a dual variational method and we determine the values of a parameter λ for which a solution to (1.1) or equivalently to (1.3) exists. The dual variational method is applied in the case where there are no global growth conditions allowing using another variational approach. Despite it relying on minimizing of the action functional $J_\lambda : R^N \to R$ given by $J_\lambda(u) = \frac{1}{2}\langle Au, u\rangle + \lambda G(u) - \lambda H(u)$, the use of the Fermat's rule is not permitted due to the fact that the set on which we minimize J_λ need not be open.

By $|\cdot|$ we mean the Euclidean norm and $\langle\cdot,\cdot\rangle$ denotes the scalar product in R^N; $\|A\|$ is the norm of a matrix A.

2. The existence of solutions

Since there exists an uniquely determined invertible symmetric matrix S such that $S^2 = A$, we may consider (having fixed $\lambda > 0$) a system

$$S^2u + g(u) = h(u) \tag{2.1}$$

with the action functional $J(u) = \frac{1}{2}\langle Su, Su\rangle + G(u) - H(u)$. System (2.1) has enough convexity for applying a dual variational methodology. We see (for example by the direct variational approach) that the system

$$S^2v + g(v) = h(u)$$

has a solution $v \in R^N$ for any fixed $u \in R^N$. Indeed, the functional $J_u(v) = \frac{1}{2}\langle Sv, Sv\rangle + G(v) - H(u)$ is convex, coercive and differentiable.

Theorem 2.1. *We assume* A1 *and let $X \subset R^N$ be such a set that there exists $u \in X$ satisfying $J(u) = \inf_{z\in X} J(z)$. If equation $S^2v + g(v) = h(u)$ has a solution in X, then u solves (2.1).*

Proof. We may define $p, q \in R^N$ by $p = Sv$, $q = g(v)$. We have now the following two relations, of which the former follows by definition and the latter by the convexity of G and the properties of the Fenchel–Young transform (see [8]):

$$Sp + q = h(u), \tag{2.2}$$

$$\langle v, Sp\rangle - \frac{1}{2}\langle Sv, Sv\rangle + \langle v, q\rangle - G(v) = \frac{1}{2}\langle p, p\rangle + G^*(q). \tag{2.3}$$

By the Fenchel–Young inequality

$$\langle z, Sp\rangle - \frac{1}{2}\langle Sz, Sz\rangle + \langle z, q\rangle - G(z) \le \frac{1}{2}\langle p, p\rangle + G^*(q) \tag{2.4}$$

valid for any $z \in R^N$, we see that by (2.3)

$$\begin{aligned}\frac{1}{2}\langle p, p\rangle + G^*(q) &\le \sup_{z\in X}\left\{\langle z, Sp\rangle - \frac{1}{2}\langle Sz, Sz\rangle + \langle z, q\rangle - G(z)\right\}\\ &\le \sup_{z\in R^N}\left\{\langle z, Sp\rangle - \frac{1}{2}\langle Sz, Sz\rangle\right\} + \sup_{z\in R^N}\{\langle z, q\rangle - G(z)\}\\ &\le \frac{1}{2}\langle p, p\rangle + G^*(q).\end{aligned} \tag{2.5}$$

By (2.2) and since $\frac{\partial H(x)}{\partial x_i} = h_i(x_i)$ we see by convexity that

$$\langle Sp + q, u\rangle = H(u) + H^*(Sp + q). \tag{2.6}$$

Now by inequality (2.4) with $z = u$ and by (2.6) we obtain

$$\begin{aligned} J(u) &\geq \langle u, Sp\rangle - \frac{1}{2}\langle p, p\rangle + \langle u, q\rangle - G^*(q) - H(u) \\ &= -\frac{1}{2}\langle p, p\rangle - G^*(q) + H^*(Sp + q). \end{aligned}$$

Next we see by inequality $-H(z) \leq -\langle Sp + q, z\rangle + H^*(Sp + q)$ and (2.5) that

$$\begin{aligned} J(u) &= \inf_{z\in X}\left\{\frac{1}{2}\langle Sz, Sz\rangle + G(z) - H(z)\right\} \\ &\leq \inf_{z\in X}\left\{-\langle p, Sz\rangle + \frac{1}{2}\langle Sz, Sz\rangle - \langle q, z\rangle + G(z)\right\} + H^*(Sp + q) \\ &= -\frac{1}{2}\langle p, p\rangle - G^*(q) + H^*(Sp). \end{aligned}$$

Hence $\frac{1}{2}\langle Su, Su\rangle + G(u) - H(u) = -\frac{1}{2}\langle p, p\rangle - G^*(q) + H^*(Sp)$ and by (2.6) we see that

$$\frac{1}{2}\langle Su, Su\rangle + \frac{1}{2}\langle p, p\rangle - \langle p, Su\rangle + G(u) + G^*(q) - \langle q, u\rangle = 0.$$

By the Fenchel–Young inequalities $\frac{1}{2}\langle Su, Su\rangle + \frac{1}{2}\langle p, p\rangle - \langle p, Su\rangle \geq 0$ and $G(u) + G^*(q) - \langle q, u\rangle = 0 \geq 0$ we have

$$\begin{aligned} &\frac{1}{2}\langle Su, Su\rangle + \frac{1}{2}\langle p, p\rangle - \langle p, Su\rangle = 0, \\ &G(u) + G^*(q) - \langle q, u\rangle = 0. \end{aligned}$$

Hence, we get $Su = p$ and $g(u) = q$. Thus by (2.2) we see that (2.1) is satisfied. □

3. Applications of the existence result

Suppose that $M_d = \max_{z\in[-d,d]} |h(z)|$ for $d > 0$. Since A is positive definite there exists a constant $m > 0$ such that $m|v|^2 \leq \langle Av, v\rangle$ for all $v \in R^N$. We proceed with two applications of our general existence result for system (1.3) which is in turn equivalent to system (1.1).

Theorem 3.1. *Assume* A1. *Let us fix $d > 0$. For each $0 < \lambda \leq \frac{md}{M_d}$ there exists a solution*

$$u \in X = \{z \in R^N : |z| \leq d\}.$$

to (1.3) and thus to (1.1). Suppose additionally that $f(0) \neq 0$. Then, for each $0 < \lambda \leq \frac{md}{M_d}$ there exists a nontrivial solution $u \in R^N$ to (1.1) such that $J_\lambda(u) = \inf_{v\in X} J_\lambda(v)$.

Proof. At least one argument of a minimum for J_λ over X obviously exists, which we denote by u. There exists a solution $v \in R^N$ to the problem

$$Av + \lambda g(v) = \lambda h(u). \tag{3.1}$$

We see by the definition of g that $\lambda g(v)v \geq 0$. Thus multiplying (3.1) by v we further obtain

$$m|v|^2 \leq \langle Av, v\rangle + \lambda g(v)v = \lambda\langle h(u), v\rangle \leq \lambda M_d|v|.$$

Since $\lambda \leq \frac{md}{M_d}$ we have $|v| \leq \frac{\lambda M_d}{m} \leq d$ and $v \in X$. The result now follows by Theorem 2.1. □

Theorem 3.2. *Assume* A1. *Let there exist numbers $c, \alpha > 0$ such that*

$$|f(z)| \geq \alpha \quad \textit{for } |z| \geq c. \tag{3.2}$$

Let us fix $d > c > 0$. Suppose that $|h(z)| < \frac{md\alpha}{\|A\|c}$ for any $|z| \leq d$. For each $\lambda \in [\frac{\|A\|c}{\alpha}, \frac{md}{M_d}]$ system (1.3) has a solution $u \in X_1 = \{z \in R^N : c \leq |z| \leq d\}$; u also solves (1.1).

Proof. As in Theorem 3.1 we obtain that for each $z \in X_1$ there exists a solution $v \in R^N$ to $Av + \lambda g(v) = \lambda h(z)$ such that $|v| \leq d$. Next, we see by (3.2) and by $\lambda \geq \frac{\|A\|c}{\alpha}$ that $|v| \geq \frac{|Av|}{\|A\|} = \frac{\lambda|h(z)|}{\|A\|} \geq \frac{\alpha\lambda}{\|A\|} \geq c$. So $v \in X_1$ and the assertion follows by Theorem 2.1. □

4. Examples and some generalization

Superlinear functions $f_i(x) = 2xe^{x^2} - e^x$ and $g_i(x) = \frac{1}{p}|x|^{p-2}x - x^2$ with $p > 2$ and $i = 1, 2, \ldots, N$ satisfy the assumptions of Theorem 3.1 while they do not satisfy the assumptions given in [4] and due to their nonconvexity also they may not be treated by the method from [7].

The method which we have just described can also be applied in the case of making the following assumption instead of A1:

A2 $f_i : R \to R$ for $i = 1, 2, \ldots, N$ is a continuous function such that for any $d > 0$ there exists a constant $\alpha_d > 0$ with the property that function $x \to \sum_{i=1}^{N} F_i(x_i) + \frac{1}{2} \sum_{i=1}^{N} \alpha_d x_i^2$ is convex in $[-d, d]$.
Assumption A2 seems weaker than A1. Both Theorems 3.1 and 3.2 hold with A1 weaker than A2. We may also replace A1 in Theorems 3.1 and 3.2 with the following assumption which is stronger than A1 but readily verified:

A3 $f_i : R \to R$ for $i = 1, 2, \ldots, N$ is a continuous function such that $f_i = h_i - g_i$ for $i = 1, 2, \ldots, N$, where h_i, g_i are convex nondecreasing functions and $g(x)x \geq 0$ for $x \in R^N$.
The functions mentioned at the beginning satisfy A1, A2 and do not satisfy A3. Function $h_i(x) = \frac{1}{p}|x|^{p-2}x - x^3$ for $i = 1, 2, \ldots, N$ satisfies A3 and, furthermore, A1 and A2.

References

[1] X. Cai, Z. Guo, Existence of solutions of nonlinear fourth order discrete boundary value problem, J. Difference Equ. Appl. 12 (5) (2006) 459–466.
[2] Y. Li, The existence of solutions for second-order difference equations, J. Difference Equ. Appl. 12 (2) (2006) 209–212.
[3] Y. Tian, Z. Du, W. Ge, Existence results for discrete Sturm–Liouville problem via variational methods, J. Difference Equ. Appl. 13 (6) (2007) 467–478.
[4] Y. Yang, Z. Zhang, Existence results for a nonlinear system with a parameter, J. Math. Anal. Appl. 340 (1) (2008) 658–668.
[5] G. Zhang, S.S. Cheng, Existence of solutions for a nonlinear system with a parameter, J. Math. Anal. Appl. 314 (1) (2006) 311–319.
[6] G. Zhang, Existence of non-zero solutions for a nonlinear system with a parameter, Nonlinear Anal. Theory Methods Appl. 66 (6) (2007) 1410–1416. (A).
[7] M. Galewski, A note on the existence of a bounded solution for a nonlinear system of difference equations, J. Difference Equ. Appl. 16 (1) (2010) 121–124.
[8] I. Ekeland, R. Temam, Convex Analysis and Variational Problems, North-Holland, Amsterdam, 1976.